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The energy operator for a model with a multiparametric infinite statistics

Stjepan Meljanac[1], **Ante Perica**[1] **and Dragutin Svrtan**[2]

[1] Rudjer Bošković Institute – Bijenička c. 54, 10000 Zagreb, Croatia
[2] Department of Mathematics, University of Zagreb, Bijenička c. 30, 10000 Zagreb, Croatia

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Abstract

In this paper we consider an energy operator (a free Hamiltonian), in the second-quantized approach, for the multiparameter quon algebras: $a_i a_j^\dagger - q_{ij} a_j^\dagger a_i = \delta_{ij}, i, j \in I$ with $(q_{ij})_{i,j\in I}$ any Hermitian matrix of deformation parameters. We obtain an elegant formula for normally ordered (sometimes called Wick-ordered) series expansions of number operators (which determine a free Hamiltonian). As a main result (see theorem 1) we prove that the number operators are given, with respect to a basis formed by 'generalized Lie elements', by certain normally ordered quadratic expressions with coefficients given precisely by the entries of the inverses of Gram matrices of multiparticle weight spaces. (This settles a conjecture by Meljanac S and Perica A (1994 *J. Phys. A: Math. Gen.* **27** 4737–44).) These Gram matrices are Hermitian generalizations of the Varchenko matrices, associated with a quantum (symmetric) bilinear form of diagonal arrangements of hyperplanes. The solution of the inversion problem of such matrices in Meljanac S and Svrtan D (1996 *Math. Commun.* **1** 1–24 (theorem 2.2.17)), leads to an effective formula for the number operators studied in this paper. The one-parameter case, in the monomial basis, was studied by Zagier, Stanciu and Møller.

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1. Introduction

One-parameter quonic intermediate statistics [2–4], which interpolate between Bose–Einstein and Fermi–Dirac statistics, are examples of infinite statistics in which any representation of the symmetric group can occur. These models offer a possibility of a small violation of the Pauli exclusion principle, at least in nonrelativistic theory [3, 5]. In a seminal paper [15], Zagier made an explicit computation of the Gram determinants of multiparticle weight spaces of the Fock representation (which for $q \in \langle -1, 1\rangle$ proves a Hilbert space realizability of 'q-mutator relations' $a_i a_j^\dagger - q a_j^\dagger a_i = \delta_{ij}, i, j \in I$) and began a study of particle number operators.

A slight variation of the Zagier conjecture [15] on the form of a normally ordered series expansion of the number operators in a monomial basis was proved subsequently by Stanciu in [11]. Generally, physical observables in the second-quantized approach are represented in terms of creation and annihilation operators in the normally ordered form (see Møller [6]). Meljanac and Perica started (in [7, 8]) with an idea to extend the above results to the multiparameter case: $a_i a_j^\dagger - q_{ij} a_j^\dagger a_i = \delta_{ij}$, $i, j \in I$, where each commutation relation has its own deformation parameter q_{ij} (a complex number) satisfying $q_{ji} = (q_{ij})^*$ (where '$*$' denotes complex conjugation).

Subsequently, in [9] (see also [10]) two types of results are proved:

Ad.1. In the case of distinct quantum numbers the multiparameter Gram determinants (theorem 1.9.2) are computed by extending Zagier's method, which in turn also gives a Hermitian analogue of the Varchenko determinant of the (symmetric) quantum bilinear form of diagonal arrangements of hyperplanes. From this explicit computation a Hilbert space realizability follows in the case when all $|q_{ij}| < 1$ (cf other methods presented in [16, 17]).

Ad.2. Explicit formulae (theorem 2.2.17) are obtained for the inverse of the Gram matrices of arbitrary multiparticle weight spaces, by following the ideas of Božejko and Speicher (given in [16]). In particular, a counterexample (when $n = 8$) to a conjecture of Zagier (also stated in [15]), for the form of the inverse in the one-parameter case, is found. In [9] an appropriate extension of Zagier's conjecture for the form of the inverse of multiparameter Gram matrices is also formulated and proved.

In this paper, we study number operators (and hence energy operator) in the spirit of the second-quantized approach. The approach is basically algebraic, i.e. independent of any particular representation (see [3, 6, 8, 11]).

The main result of this paper is theorem 1, in which we show that the coefficients of the normally ordered series expansion of particle number operators in the Fock representation, in terms of a basis of 'generalized Lie elements', are given precisely by certain inverse matrix entries of the Gram matrices on the multiparticle weight spaces. This confirms a conjecture of Meljanac and Perica in [8]. Thus, in conjunction with the results of [9], one obtains explicit expressions for the number operators in multiparameter quon algebras.

2. Multiparameter quon algebras and Gram matrices

Let $\mathbf{q} = \{q_{ij} : i, j \in I, (q_{ij})^* = q_{ji}\}$ be a Hermitian family of complex numbers (parameters), where I is a finite (or infinite) set of indices. Recall that (cf [9]) by a *multiparameter quon algebra* $\mathcal{A} = \mathcal{A}^{(\mathbf{q})}$ we mean an associative (complex) algebra generated by $\{a_i, a_i^\dagger, i \in I\}$ subject to the following q_{ij}-canonical commutation relations:

$$a_i a_j^\dagger = q_{ij} a_j^\dagger a_i + \delta_{ij} \qquad \forall i, j \in I.$$

The algebra $\mathcal{A}$ has a canonical anti-involution '$\dagger$': $\mathcal{A} \to \mathcal{A}$ (which exchanges a_i with $a_i^\dagger$, reverses products and on the coefficients acts by complex conjugation).

Recall that a Fock representation of $\mathcal{A}$ is given by a family of linear operators $a_i : \mathcal{H} \to \mathcal{H}$ on a complex Hilbert space $\mathcal{H}$, $i \in I$, satisfying the following canonical commutation (or 'q_{ij}-mutator') relations,

$$a_i a_j^\dagger - q_{ij} a_j^\dagger a_i = \delta_{ij} \qquad i, j \in I \tag{1}$$

$$a_i |0\rangle = 0 \qquad i \in I \tag{2}$$

where $a_i^\dagger$ denotes the adjoint of a_i, and $|0\rangle$ denotes a distinguished ('vacuum') vector in $\mathcal{H}$.

Any total order on the indexing set I induces a total order on the set I^* of all sequences (=words) $\mathbf{i} = i_1 \cdots i_n$ over I. Then we can consider the Gram matrix

$$A = (\langle \mathbf{i} | \mathbf{j} \rangle) \tag{3}$$

of all n-particle states $|\mathbf{i}\rangle := a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_n}^\dagger |0\rangle$ ($i_j \in I, n \geqslant 0$). Its entries $\langle \mathbf{i}|\mathbf{j}\rangle$ are the 'expectation values' (i.e. overlaps of n-particle states in the second quantized Fock description)

$$\langle 0 | a_{i_n} \cdots a_{i_1} a_{j_1}^\dagger \cdots a_{j_m}^\dagger |0\rangle.$$

These entries vanish, unless (i) $n = m$ and (ii) $i_1 \cdots i_n$ and $j_1 \cdots j_m$ are permutations of the same weakly increasing sequences $\nu = k_1 \ldots k_n, k_1 \leqslant \cdots \leqslant k_n, k_j \in I$, which we shall call *weights*. Thus the matrix A is block diagonal (cf [9, proposition 1.6.1]):

$$A = \oplus_{n \geqslant 0} \oplus_{k_1 \leqslant \cdots \leqslant k_n} A^{k_1 \ldots k_n} \tag{4}$$

with blocks $A^\nu = A^{k_1 \ldots k_n}$ indexed by weights. The size of A^ν is equal to the number of permutations (or rearrangements) of the multiset $\{k_1 \leqslant \cdots \leqslant k_n\}$.

For $\nu = k_1 < k_2 < \cdots < k_n$ (a generic weight), A^ν is a matrix of order $n!$ with rows/columns labelled by rearrangements (of ν) $\mathbf{i} = i_1 \cdots i_n = k_{\pi(1)} \ldots k_{\pi(n)} =: \nu \cdot \pi$ ($\pi \in S_n =$ the nth symmetric group) or simply by permutations $\pi \in S_n$. The entry of A^ν in the row $\mathbf{i} = \nu \cdot \pi$ and column $\mathbf{j} = \nu \cdot \sigma$ is then given explicitly by the following formula,

$$A^\nu_{\mathbf{i},\mathbf{j}} = A^\nu(\pi, \sigma) = \prod_{(r,s) \in I(\sigma^{-1}\pi)} q_{k_{\pi(r)} k_{\pi(s)}} \tag{5}$$

where, for $\pi \in S_n$, $I(\pi)$ denotes the set of inversions of π: $I(\pi) = \{(r, s) : 1 \leqslant r < s \leqslant n, \pi(r) > \pi(s)\}$. Thus, we can view A^ν as a linear operator on the group algebra $\mathbf{C}[S_n] = \{\sum_{\pi \in S_n} c_\pi \pi \colon c_\pi \in \mathbf{C}, \pi \in S_n\}$.

For general weights $\tilde{\nu} = \big(\tilde{k}_1 = \cdots = \tilde{k}_{n_1} < \tilde{k}_{n_1+1} = \cdots = \tilde{k}_{n_1+n_2} < \cdots < \tilde{k}_{n_1+\cdots+n_{p-1}+1} = \cdots = \tilde{k}_n\big), n_1 + n_2 + \cdots + n_p = n$, the matrix $A^{\tilde{\nu}}$ has order equal to $n!/n_1! \cdots n_p!$ and its rows/columns are labelled by rearrangements $\mathbf{i} = i_1 \cdots i_n = \tilde{\nu} \cdot \tilde{\pi}, \tilde{\pi} \in H_{\tilde{\nu}} \backslash S_n$, where $H_{\tilde{\nu}} = \mathrm{Stab}_{\tilde{\nu}} = \{\sigma \in S_n | \tilde{\nu} \cdot \sigma = \tilde{\nu}\}$ is the (stabilizer) subgroup fixing $\tilde{\nu}$. The $(\mathbf{i}, \mathbf{j})$th entry of $A^{\tilde{\nu}}$, $\mathbf{i} = \tilde{\nu} \cdot \tilde{\pi}, \mathbf{j} = \tilde{\nu} \cdot \tilde{\sigma}, \tilde{\pi} = H\pi, \tilde{\sigma} = H\sigma$, where π, σ are unique coset representatives (of minimal length) of $\tilde{\pi}, \tilde{\sigma}$, is given by

$$A^{\tilde{\nu}}_{\mathbf{i},\mathbf{j}} = A^{\tilde{\nu}}(\tilde{\pi}, \tilde{\sigma}) = \sum_{\tau \in \tilde{\sigma}^{-1}\tilde{\pi} = \sigma^{-1} H \pi} \prod_{(r,s) \in I(\tau)} q_{i_r i_s} = \sum_{\tau \in \sigma^{-1} H \pi} \prod_{(r,s) \in I(\tau)} q_{k_{\pi(r)} k_{\pi(s)}}. \tag{$\tilde{5}$}$$

(Note that ($\tilde{5}$) generalizes (5), because $\mathrm{Stab}_\nu = H_\nu = \{1\}$, if ν is generic.) In [9, subsection 1.7] it is shown that the operator $A^{\tilde{\nu}}$ can be obtained from $A^\nu (\nu = k_1 < \cdots < k_n)$ by a *reduction procedure* in two steps: first by identifying indices $k_1 \mapsto \tilde{k}_1, \ldots, k_n \mapsto \tilde{k}_n$ and then restricting this specialized operator $A^\nu|_{\nu \mapsto \tilde{\nu}}$ to the invariant subspace (in $\mathbf{C}[S_n]$) spanned by $H_{\tilde{\nu}}$-invariant vectors $\overline{\sigma} = \sum_{h \in H_{\tilde{\nu}}} h\sigma \in \mathbf{C}[S_n]$. In fact ($\tilde{5}$) can be rewritten as

$$A^{\tilde{\nu}}(\tilde{\pi}, \tilde{\sigma}) = \sum_{h \in H_{\tilde{\nu}}} A^\nu(\pi, h\sigma)|_{\nu \mapsto \tilde{\nu}}. \tag{6}$$

As a consequence we obtain the following: if $A^\nu|_{\nu \mapsto \tilde{\nu}}$ is invertible, then the matrix $A^{\tilde{\nu}}$ is invertible too, and a relation analogous to (6) holds for the inverses. In particular, $\det A^{\tilde{\nu}}$ divides $\det A^\nu|_{\nu \mapsto \tilde{\nu}}$. This shows that in order to study some properties (e.g. invertibility or positive definiteness) it suffices to consider the generic case (when all the indices k_i are distinct).

Now we list some properties of the matrices A^ν, $\nu = k_1 < k_2 \cdots < k_n$:

$$\text{(a)} \quad A^\nu(\pi, \pi) = 1 \tag{7}$$

$$\text{(b)}\quad A^\nu(\sigma,\pi) = A^\nu(\pi,\sigma)^* \quad \text{(Hermiticity)} \tag{8}$$

(c) Let $w_n = n\ldots 21$ be the longest permutation in S_n. Then

$$A^\nu(\pi w_n, \sigma w_n) = A^\nu(\sigma,\pi) = A^\nu(\pi,\sigma)^*. \tag{9}$$

Property (c) can be rewritten in the matrix form as follows,

$$WA^\nu W = (A^\nu)^T \qquad W^2 = 1 \tag{10}$$

where

$$W(\pi,\sigma) = \begin{cases} 1 & \text{if} \quad \pi w_n = \sigma \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

It is important to note that the Fock space, in our case, is positive definite iff the Gram matrix A is positive definite. Recall that a sufficient condition for the positivity of the norm squared of all vectors is (cf [9, theorem 1.9.4])

$$|q_{ij}| < 1 \qquad \forall i, j \in I. \tag{12}$$

In particular, condition (12) implies that the n-particle states $|\mathbf{i}\rangle = a^\dagger_{i_1}\cdots a^\dagger_{i_n}|0\rangle$ $(i_j \in I, n \geqslant 0)$ are linearly independent.

Examples. For the generic weights $\nu = 1, 12, 123$ the Gram matrices are as follows:

$$A^1 = (1) \qquad A^{12} = \begin{pmatrix} 1 & q_{12} \\ q_{21} & 1 \end{pmatrix}$$

$A^{123} =$

$\pi \diagdown \sigma$	123	132	312	321	231	213
123	1	q_{23}	$q_{13}q_{23}$	$q_{12}q_{13}q_{23}$	$q_{12}q_{13}$	q_{12}
132	q_{32}	1	q_{13}	$q_{12}q_{13}$	$q_{12}q_{13}q_{32}$	$q_{12}q_{32}$
312	$q_{31}q_{32}$	q_{31}	1	q_{12}	$q_{12}q_{32}$	$q_{12}q_{31}q_{32}$
321	$q_{21}q_{31}q_{32}$	$q_{21}q_{31}$	q_{21}	1	q_{32}	$q_{31}q_{32}$
231	$q_{21}q_{31}$	$q_{21}q_{31}q_{23}$	$q_{21}q_{23}$	q_{23}	1	q_{31}
213	q_{21}	$q_{21}q_{23}$	$q_{21}q_{13}q_{23}$	$q_{13}q_{23}$	q_{13}	1

(Here we use the Johnson–Trotter ordering of permutations: 123, 132, 312, 321, 231, 213.)

For the non-generic: $\tilde{\nu} = 11, 113$, the Gram matrices are

$A^{11} = (1 + q_{11}) \qquad A^{113} =$

$\pi \diagdown \sigma$	113	131	311
113	$1+q_{11}$	$q_{13}+q_{11}q_{13}$	$q_{13}^2+q_{11}q_{13}^2$
131	$q_{31}+q_{11}q_{31}$	$1+q_{11}q_{13}q_{31}$	$q_{13}+q_{11}q_{13}$
311	$q_{31}^2+q_{11}q_{31}^2$	$q_{31}+q_{11}q_{31}$	$1+q_{11}$

The inverses of the Gram matrices in the generic case above are given by

$$[A^{12}]^{-1} = \frac{1}{\Delta^{12}}\begin{pmatrix} 1 & -q_{12} \\ -q_{21} & 1 \end{pmatrix} = \frac{1}{\Delta^{12}}\begin{pmatrix} 1 & q_{12} \\ q_{21} & 1 \end{pmatrix} * \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

where $\Delta^{12} := 1 - q_{12}q_{21} = 1 - |q_{12}|^2$, and

$$[A^{123}]^{-1} = \frac{1}{\Delta^{123}} A^{123} * M^{123}.$$

Here $\Delta^{123} := (1-|q_{12}|^2)(1-|q_{13}|^2)(1-|q_{23}|^2)(1-|q_{12}|^2|q_{13}|^2|q_{23}|^2)$, '$*$' denotes the Schur product of matrices $(a_{ij}) * (b_{ij}) := (a_{ij}b_{ij})$ and M^{123} stands for the following matrix,

$\pi \diagdown \sigma$	123	132	312	321	231	213
123	$(1-ac)(1-b)$	$(b-1)(1-c)$	$c(b-1)(1-a)$	$(1-ac)(1-b)$	$a(b-1)(1-c)$	$(b-1)(1-a)$
132	$(c-1)(1-b)$	$(1-ab)(1-c)$	$(c-1)(1-a)$	$a(c-1)(1-b)$	$(1-ab)(1-c)$	$b(c-1)(1-a)$
312	$(a-1)(1-b)$	$(a-1)(1-c)$	$(1-bc)(1-a)$	$(a-1)(1-b)$	$b(a-1)(1-c)$	$(1-bc)(1-a)$
321	$(1-ac)(1-b)$	$a(b-1)(1-c)$	$(b-1)(1-a)$	$(1-ac)(1-b)$	$(b-1)(1-c)$	$c(b-1)(1-a)$
231	$a(c-1)(1-b)$	$(1-ab)(1-c)$	$b(c-1)(1-a)$	$(c-1)(1-b)$	$(1-ab)(1-c)$	$(c-1)(1-a)$
213	$(a-1)(1-b)$	$b(a-1)(1-c)$	$(1-bc)(1-a)$	$c(a-1)(1-b)$	$(a-1)(1-c)$	$(1-bc)(1-a)$

(with $a := |q_{23}|^2, b := |q_{13}|^2, c := |q_{12}|^2$).

The inverse in the non-generic case $\nu = 113$ is given by

$$[A^{113}]^{-1} = \frac{1}{\Delta^{113}} \begin{pmatrix} 1 & -(1+q_{11})q_{13} & q_{11}q_{13}^2 \\ -q_{31}(1+q_{11}) & (1+q_{11})(1+q_{13}q_{31}) & -(1+q_{11})q_{13} \\ q_{31}^2 q_{11} & -q_{31}(1+q_{11}) & 1 \end{pmatrix}$$

where $\Delta^{113} = (1+q_{11})(1-q_{13}q_{31})(1-q_{11}q_{13}q_{31}) = (1+q_{11})(1-|q_{13}|^2)(1-q_{11}|q_{13}|^2)$.

3. Series expansions of number operators

First we recall that the kth particle number operator N_k $(k \in I)$ (in the Fock representation satisfying the positivity condition (12)) is a diagonal operator which counts the number of appearances of the creation operator $a_k^\dagger$ in any multiparticle state $|\mathbf{i}\rangle$. These operators satisfy the following implicit conditions (equations):

$$\begin{aligned} &[N_k, a_l] = -a_k\delta_{kl} && \forall k, l \in I \\ &N_k|0\rangle = 0 && \forall k \in I. \end{aligned} \tag{13}$$

Note that for any fixed $k \in I$, if we assume (12), equations (13) have a unique solution for N_k. The number operators play an important role in constructing the free Hamiltonian (=the energy operator) of the free system (for which the energy is additive, cf [3]) of generalized quon particles in the nonrelativistic limit:

$$H = \sum_{k\in I} E_k N_k. \tag{14}$$

More generally, our primary goal here is to express N_k in terms of quon algebra generators as a normally ordered infinite series involving certain iterated deformed commutators of the creation and annihilation operators.

It is already indicated in [8] that the formal expansion of the number operator N_k in terms of normally ordered products is necessarily of the following form which preserves each n-particle subspace (it easily follows from (3)),

$$N_k = \sum_{\mathbf{i}\in I^+, i_1=k} X_{\mathbf{i}}^\dagger Y_{\mathbf{i}} \tag{15}$$

where I^+ denotes the set of all nonempty words (or sequences) $\mathbf{i} = i_1 \ldots i_n, n \geqslant 1$ over the set I as an alphabet, and the sum is over those words which begin with letter k. Here, if the indices $i_1, \ldots, i_n$ are distinct, we require that $X_{\mathbf{i}}$ and $Y_{\mathbf{i}}$ are both multihomogeneous of the same multidegree, i.e. they are expressible as a linear combination of all rearrangements $a_{\mathbf{j}} = a_{\mathbf{i}} \cdot \pi := a_{\mathbf{i}\cdot\pi}$ $(=a_{i_{\pi(1)}} a_{i_{\pi(2)}} \cdots a_{i_{\pi(n)}})$ of the 'monomial' $a_{\mathbf{i}} = a_{i_1} a_{i_2} \cdots a_{i_n}$, in the following form,

$$X_{\mathbf{i}} = \sum_{\pi \in S_n} a_{\mathbf{i}\cdot\pi} x_{\mathbf{i}\cdot\pi,\mathbf{i}} \tag{16}$$

$$Y_{\mathbf{i}} = \sum_{\pi \in S_n} a_{\mathbf{i}\cdot\pi} y_{\mathbf{i}\cdot\pi,\mathbf{i}} \tag{17}$$

where $x_{\mathbf{i}\cdot\pi,\mathbf{i}}$ and $y_{\mathbf{i}\cdot\pi,\mathbf{i}}$ are, as yet unknown, coefficients (depending on q_{ij}) with the following normalization convention $y_{\mathbf{i},\mathbf{i}} = 1$. For general $\mathbf{i}$, we require that the summations in (16) and (17) should be replaced by summations over the left cosets $H\backslash S_n$, where $H = \mathrm{Stab}_{\mathbf{i}}$ is the stabilizer subgroup of S_n fixing $\mathbf{i}$, with coefficients $\tilde{x}_{\mathbf{i}\cdot\tilde{\pi},\mathbf{i}}$, $\tilde{y}_{\mathbf{i}\cdot\tilde{\pi},\mathbf{i}}$, $\tilde{\pi} \in H\backslash S_n$ equal to the following orbit sums:

$$\begin{aligned}\tilde{x}_{\mathbf{i}\cdot\tilde{\pi},\mathbf{i}} &= \sum_{h\in H} x_{\mathbf{i}\cdot h\pi,\mathbf{i}}\\ \tilde{y}_{\mathbf{i}\cdot\tilde{\pi},\mathbf{i}} &= \sum_{h\in H} y_{\mathbf{i}\cdot h\pi,\mathbf{i}}.\end{aligned} \tag{18}$$

Now we start finding the solution of the system (13), in the form (15), as follows: we first use the fact that under condition (12), the set of all monomials $a^{\dagger}_{i_n}\cdots a^{\dagger}_{i_1}a_{j_1}\cdots a_{j_m}$ $(i_k, j_l \in I)$ is linearly independent. Then, we plug the right-hand side of (15) into the system (13). By resolving it successively in degree 1, then in degree 2, etc, we obtain the following (noncommutative) recursions for $Y_{\mathbf{i}}$:

Recursions for Y

$$Y_{i_1 i_2\cdots i_n} = Y_{i_1\cdots i_{n-1}}a_{i_n} - q_{i_n i_1}q_{i_n i_2}\cdots q_{i_n i_{n-1}}a_{i_n}Y_{i_1\cdots i_{n-1}} \tag{19}$$

and similarly, a system of 'twisted' partial differential equations for $X_{\mathbf{i}}$,

Equations for X

$${}_l\partial\big(X_{i_1\cdots i_n}\big)^{\dagger} = \big(X_{i_1\cdots i_{n-1}}\big)^{\dagger}\delta_{l i_n} \qquad (l \in \{i_1,\ldots,i_n\}) \tag{20}$$

where ${}_l\partial$ denotes the left twisted derivative,

$${}_l\partial\big(a^{\dagger}_{j_1}\cdots a^{\dagger}_{j_n}\big) = \sum_{(p:j_p=l)} q_{lj_1}\cdots q_{lj_{p-1}}a^{\dagger}_{j_1}\cdots\widehat{a^{\dagger}_{j_p}}\cdots a^{\dagger}_{j_n} \tag{21}$$

($\widehat{\ }$ denotes the omission of the corresponding creation operator).

Proposition 1. *The Y-components (17) of the solution (15) of equation (13) are given by the following iterated* $\mathbf{q}$*-commutator ('generalized Lie elements') formula,*

$$\begin{aligned}Y_{i_1} &= a_{i_1}\\ Y_{i_1 i_2\ldots i_n} &= \big[\cdots\big[\big[a_{i_1},a_{i_2}\big]_{q_{i_2 i_1}},a_{i_3}\big]_{q_{i_3 i_1}q_{i_3 i_2}},\ldots,a_{i_n}\big]_{q_{i_n i_1}q_{i_n i_2}\cdots q_{i_n i_{n-1}}}\end{aligned} \tag{22}$$

where $[x,y]_q = xy - qyx$ *denotes the q-commutator of x and y. (For* N_k *we need to set* $i_1 = k$*.)*

Proof. By iterating (19). □

In order to express the formula (22) (and some others later) in the operator form we shall now introduce a twisted group algebra of the permutation group.

4. A twisted group algebra action

Let us consider the following.

(1) A *right action* of the symmetric group S_n, by permuting factors of any degree n monomial in the annihilation operators:

$$a_{\mathbf{i}} \cdot \pi = \left(a_{i_1} a_{i_2} \cdots a_{i_n}\right) \cdot \pi := a_{i_{\pi(1)}} a_{i_{\pi(2)}} \cdots a_{i_{\pi(n)}} \tag{23}$$

(2) A '*diagonal' action* of the formal power series ring $K_n = \mathbf{C}[[Q_{k,l}, 1 \leqslant k, l \leqslant n]]$ (where $Q_{k,l}$ are commuting indeterminates) defined by

$$a_{\mathbf{i}} \cdot Q_{k,l} \left(=\left(a_{i_1} a_{i_2} \cdots a_{i_n}\right) \cdot Q_{k,l}\right) := q_{i_k i_l} a_{i_1} a_{i_2} \cdots a_{i_n}. \tag{24}$$

(Here q_{ij} are complex numbers from the canonical commutation relations (1)!) These two actions give rise to an action of a *twisted group algebra*:

$$\mathcal{K}_n = K_n\tilde{\ }[S_n] \tag{25}$$

of S_n (with coefficients in K_n). The multiplication in the algebra $\mathcal{K}_n$ is defined by imposing the following commutation relations ('an action of S_n on the coefficient ring K_n')

$$\pi Q_{k,l} = Q_{\pi(k)\pi(l)}\pi. \tag{26}$$

It is clear that, by specializing $Q_{k,l} = q$ $(1 \leqslant k, l \leqslant n)$, the twisted group algebra $K_n\tilde{\ }[S_n]$ is mapped onto the ordinary group algebra $\mathbf{C}[[q]][S_n]$ in which, according to Zagier [15], live certain important elements: $\alpha_n, \beta_n, \gamma_n, \delta_n$ satisfying

$$\alpha_n = \alpha_{n-1}\beta_n, \beta_n = \delta_n {\gamma_n}^{-1} \left(\Rightarrow \alpha_n = \beta_2 \cdots \beta_n = \delta_2 \gamma_2^{-1} \delta_3 \gamma_3^{-1} \cdots \gamma_{n-1}^{-1} \delta_n \gamma_n^{-1}\right). \tag{27}$$

(Note that our notation for δ_n is shifted by 1 compared with [15], which seems to be more natural!)

These elements, via the regular representation R_n, were crucial in Zagier's computation of the determinant and the inverse of the one-parameter matrices $A_n = A_n(q) = R_n(\alpha_n)$. We shall now define a 'lifting' to $K_n\tilde{\ }[S_n]$ of the Zagier elements by first defining, for each permutation $\pi \in S_n$, an element $\tilde{\pi} \in K_n\tilde{\ }[S_n]$, $(\pi \in S_n)$, which encodes all inversions of π:

$$\tilde{\pi} := Q_\pi \pi \qquad \text{where} \quad Q_\pi := \prod_{1 \leqslant k < l \leqslant n, \pi(k) > \pi(l)} Q_{\pi(k),\pi(l)} \tag{28}$$

with the multiplication rule

$$\tilde{\sigma}\tilde{\pi} = \left(\prod_{(a,b) \in I(\sigma) \cap I(\pi^{-1})} Q_{\sigma(a),\sigma(b)} Q_{\sigma(b),\sigma(a)} \right) \widetilde{\sigma\pi}.$$

(Observe that $\tilde{\pi}$ generalizes $q^{i(\pi)}\pi$, $i(\pi) :=$ the number of inversions of π.)

Then we define a 'lifting' of all Zagier's elements by the following formulae:

$$\tilde{\alpha}_n := \sum_{\pi \in S_n} \tilde{\pi} \tag{29}$$

$$\tilde{\beta}_n := \sum_{k=1}^{n} \tilde{t}_{k,n} \tag{30}$$

$$\tilde{\gamma}_n := (1 - \tilde{t}_{1,n})(1 - \tilde{t}_{2,n}) \cdots (1 - \tilde{t}_{n-1,n}) \tag{31}$$

$$\tilde{\delta}_n := (1 - \tilde{t}_{n-1}\tilde{t}_{1,n})(1 - \tilde{t}_{n-1}\tilde{t}_{2,n}) \cdots (1 - \tilde{t}_{n-1}\tilde{t}_{n-1,n}). \tag{32}$$

Similarly we define

$$\tilde{\alpha}_{n_1,n_2,\ldots,n_k} := \sum_{\pi\in S_{n_1}\times S_{n_2}\times\cdots\times S_{n_k}} \tilde{\pi}. \tag{29a}$$

(Here $t_{k,l}$ denotes the cycle $\begin{pmatrix} k & k+1 & \cdots & l \\ l & k & \cdots & l-1 \end{pmatrix} \in S_n$ and $t_k := t_{k,k+1}$.)

It is easy to check that the following relations, analogous to (27), hold true:

$$\tilde{\alpha}_n = \tilde{\alpha}_{n-1}\tilde{\beta}_n,\ \tilde{\beta}_n = \tilde{\delta}_n\tilde{\gamma}_n^{-1}\ \left(\Rightarrow \tilde{\alpha}_n = \tilde{\beta}_2\cdots\tilde{\beta}_n = \tilde{\delta}_2\tilde{\gamma}_2^{-1}\tilde{\delta}_3\tilde{\gamma}_3^{-1}\cdots\tilde{\gamma}_{n-1}^{-1}\tilde{\delta}_n\tilde{\gamma}_n{}^{-1}\right). \tag{33}$$

Important note. Now we can realize all Gram matrices A^ν from (4) as the matrices of the right multiplication by the lifted Zagier element $\tilde{\alpha}_n$ on the space of monomials $a_{\mathbf{i}}$ of weight ν. This explains why we needed to introduce a twisted group algebra in the multiparameter case.

In what follows, we shall also need the following notation:

$$Q_{\{\pi\}} := \prod_{1\leqslant k<l\leqslant n,\pi(k)>\pi(l)} Q_{\pi(k),\pi(l)}Q_{\pi(l),\pi(k)}\ (\text{for any } \pi\in S_n) \tag{34}$$

$$Q_T := \prod_{k\neq l\in T} Q_{k,l}\ (\text{for any set } T\subseteq\{1,2,\ldots,n\}) \tag{35}$$

together with the following lemma which we shall use in the proof of the main result:

Lemma 1. *We have the following identity in $\mathcal{K}_n$:*

$$\tilde{\alpha}_{n-1,1}(1-\tilde{t}_{n-1}\tilde{t}_{1,n}) = \xi_n\tilde{\alpha}_{1,n-2,1} \tag{36}$$

where $\xi_n := \sum_{k=1}^{n-1}(1-Q_{\{k,k+1\}}\cdots Q_{\{k,n\}})\tilde{t}_{1,k}$.
(Recall from (29a) that $\tilde{\alpha}_{n-1,1} = \sum_{\pi\in S_{n-1}\times S_1}\tilde{\pi}$, $\tilde{\alpha}_{1,n-2,1} = \sum_{\pi\in S_1\times S_{n-2}\times S_1}\tilde{\pi}$.*)*

Proof. By definition $\tilde{\alpha}_{n-1,1} = \sum_{\pi\in S_{n-1}\times S_1}\tilde{\pi}$. By using a factorization $\pi = t_{1,k}\sigma$, where $\pi(1)=k, \sigma\in S_1\times S_{n-2}\times S_1$, we get $\tilde{\alpha}_{n-1,1} = \left(\sum_{k=1}^{n-1}\tilde{t}_{1,k}\right)\tilde{\alpha}_{1,n-2,1}$ (here we used that $\tilde{\pi} = \tilde{t}_{1,k}\tilde{\sigma}$, cf (28)). Similarly,

$$\begin{aligned}
\tilde{\alpha}_{n-1,1}\tilde{t}_{n-1}\tilde{t}_{1,n} &= \sum_{\pi\in S_{n-1}\times S_1}\tilde{\pi}\tilde{t}_{n-1}\tilde{t}_{1,n} = \sum_{\pi\in S_{n-1}\times S_1}\tilde{\pi}\,Q_{\{n-1,n\}}\tilde{t}_{1,n-1} \qquad \text{(by (28))}\\
&= \sum_{\pi\in S_{n-1}\times S_1} Q_{\{\pi(n-1),\pi(n)\}}\tilde{\pi}\tilde{t}_{1,n-1}\\
&= \sum_{\pi\in S_{n-1}\times S_1} Q_{\{\pi(n-1),n\}}Q_{\{t^{-1}_{\pi(n-1),n-1}\}}\widetilde{\pi t}_{1,n-1} \qquad \text{(by (28) and (34))}\\
&= \sum_{\sigma\in S_1\times S_{n-2}\times S_1} Q_{\{t^{-1}_{\pi(n-1),n}\}}\tilde{t}_{1,\pi(n-1)}\tilde{\sigma}[t_{1,\pi(n-1)}\sigma = \pi t_{1,n-1}]\\
&= \left(\sum_{k=1}^{n-1} Q_{\{t^{-1}_{k,n}\}}\tilde{t}_{1,k}\right)\tilde{\alpha}_{1,n-2,1}.
\end{aligned}$$

By subtracting the last two formulae, the lemma follows. □

Now we state the formula (22) in the operator form:

Corollary 1. *We have*

(i) $Y_{i_1\cdots i_n} = (a_{i_1}a_{i_2}\cdots a_{i_n})\cdot\overline{\gamma_n}$, *where* $\overline{\gamma_n} := (1-\widetilde{t}_{1,2})(1-\widetilde{t}_{1,3})\cdots(1-\widetilde{t}_{1,n}) \in \mathcal{K}_n$.

(ii) $a_{i_1}a_{i_2}\cdots a_{i_n} = Y_{i_1\cdots i_n}\cdot\overline{\gamma_n}^{-1}$, *with*

$$\overline{\gamma_n}^{-1} = \sum_{\pi\in S_n}\tilde{\pi}\cdot\prod_{\pi(i)>\pi(i+1)} Q_{\{1,\ldots,i\}}/(1-Q_{\{1,2\}})\cdots(1-Q_{\{1,\ldots,n\}}).$$

(iii) The set $\{Y_{\mathbf{i}\cdot\tilde{\pi}}|\tilde{\pi}\in H\backslash S_n\}$ $(H = Stab_{\mathbf{i}})$ *is a linearly independent set if* $|q_{i_r i_s}|<1$, $1\leqslant r\neq s\leqslant n$.

Proof. (i) Formula (22) can be rewritten as

$$Y_{i_1i_2\cdots i_n} = a_{i_1}a_{i_2}\cdots a_{i_n}\big(1-q_{i_2i_1}t_{1,2}\big)\big(1-q_{i_3i_1}q_{i_3i_2}t_{1,3}\big)\cdots\big(1-q_{i_ni_1}q_{i_ni_2}\cdots q_{i_ni_{n-1}}t_{1,n}\big).$$

By using $\widetilde{t}_{1,l} = Q_{l,1}\cdots Q_{l,l-1}t_{1,l} = t_{1,l}Q_{1,2}Q_{1,3}\cdots Q_{1,l}$ the claim (i) follows. (ii) The proof of (ii) is similar to that of proposition 2.1.1 in [10]. (iii) Follows from (ii). □

Proposition 2. *The* $Y_{\mathbf{i}}$ *satisfy the following (twisted) differential equations:*

$$\text{(i)}\ {}_l\partial\big(Y_{i_1\cdots i_n}\big)^\dagger = \sum_{(j\geqslant 2:i_j=l)} d^{(j)}_{i_1\cdots i_n}\big(Y_{i_1\cdots\hat{i}_j\cdots i_n}\big)^\dagger \qquad (n\geqslant 2)$$
$$\text{(ii)}\ {}_l\partial Y^\dagger_{i_1} = \delta_{i_1 l} \qquad (n=1) \tag{37}$$

where ${}_l\partial$ *is defined in (21), and where*

$$d^{(j)}_{i_1\cdots i_n} := q_{i_ji_{j+1}}\cdots q_{i_ji_n}\big(1-\big|q_{i_ji_1}\cdots q_{i_ji_{j-1}}\big|^2\big). \tag{38}$$

Proof. By induction. For $n=2$ we have $Y_{i_1i_2} = \big[a_{i_1},a_{i_2}\big]_{q_{i_2i_1}} = a_{i_1}a_{i_2} - q_{i_2i_1}a_{i_2}a_{i_1}$ which implies $\big(Y_{i_1i_2}\big)^\dagger = a^\dagger_{i_2}a^\dagger_{i_1} - q_{i_1i_2}a^\dagger_{i_1}a^\dagger_{i_2}$ (here we use $(q_{ij})^* = q_{ji}$). Hence

$$\begin{aligned}{}_l\partial\big(Y_{i_1i_2}\big)^\dagger &= \delta_{li_2}a^\dagger_{i_1} + \delta_{li_1}q_{li_2}a^\dagger_{i_2} - q_{i_1i_2}\big(\delta_{li_1}a^\dagger_{i_2} + \delta_{li_2}q_{li_1}a^\dagger_{i_1}\big)\\ &= \big(1-q_{i_1i_2}q_{i_2i_1}\big)a^\dagger_{i_1}\delta_{li_2} = d^{(2)}_{i_1i_2}Y^\dagger_{i_1}\delta_{li_2}.\end{aligned}$$

Now we suppose that (25) holds true for $n-1$. Then, from (19) it follows that

$$\begin{aligned}{}_l\partial\big(Y_{i_1\cdots i_n}\big)^\dagger &= {}_l\partial\big[a^\dagger_{i_n}\big(Y_{i_1\cdots i_{n-1}}\big)^\dagger - q_{i_1i_n}\cdots q_{i_{n-1}i_n}\big(Y_{i_1\cdots i_{n-1}}\big)^\dagger a^\dagger_{i_n}\big]\\
&= \delta_{li_n}\big(Y_{i_1\cdots i_{n-1}}\big)^\dagger + q_{li_n}a^\dagger_{i_n}{}_l\partial\big(Y_{i_1\cdots i_{n-1}}\big)^\dagger\\
&\quad -q_{i_1i_n}\cdots q_{i_{n-1}i_n}\big[{}_l\partial\big(Y_{i_1\cdots i_{n-1}}\big)^\dagger a^\dagger_{i_n} + q_{i_ni_1}\cdots q_{i_ni_{n-1}}\delta_{li_n}\big(Y_{i_1\cdots i_{n-1}}\big)^\dagger\big]\\
&= \delta_{li_n}\big(1-\big|q_{i_1i_2}\cdots q_{i_{n-1}i_n}\big|^2\big)\big(Y_{i_1\cdots i_{n-1}}\big)^\dagger +\\
&\quad \sum_{j=2;i_j=l}^{n-1} q_{li_n}d^{(j)}_{i_1\cdots i_{n-1}}\big[a^\dagger_{i_n}\big(Y_{i_1\cdots\widehat{i_j}\cdots i_{n-1}}\big)^\dagger - q_{i_1i_n}\cdots\widehat{q_{li_n}}\cdots q_{i_{n-1}i_n}\big(Y_{i_1\cdots\widehat{i_j}\cdots i_{n-1}}\big)^\dagger a^\dagger_{i_n}\big]\\
&= \delta_{li_n}d^{(n)}_{i_1\cdots i_n}\big(Y_{i_1\cdots i_{n-1}}\big)^\dagger + \sum_{j=2;i_j=l}^{n-1} d^{(j)}_{i_1\cdots i_n}\big(Y_{i_1\cdots\widehat{i_j}\cdots i_n}\big)^\dagger = \sum_{n\geqslant j\geqslant 2;i_j=l} d^{(j)}_{i_1\cdots i_n}\big(Y_{i_1\cdots\widehat{i_j}\cdots i_n}\big)^\dagger.\end{aligned}$$

This completes the proof of proposition 2. □

Now we proceed with solving (20) to get $X_{\mathbf{i}}$-components of our number operator N_k. There are two approaches.

The first approach, developed in [8], is based on an observation that in (37) the index i_1 survives in all terms of the rhs. So, we could look for $X_{\mathbf{i}}$ in the form of a linear combination of such $Y_{\mathbf{i}}$ with the first index fixed ($= k$ for N_k).

$$(X_{\mathbf{i}})^\dagger = \sum_{\mathbf{j}=\mathbf{i}\cdot\pi,\pi\in S_1\times S_{n-1}} (Y_{\mathbf{j}})^\dagger c_{\mathbf{j},\mathbf{i}}. \tag{39}$$

By applying the twisted derivative ${}_l\partial$ to (39), the left-hand side gives

$$\begin{aligned} {}_l\partial(X_{\mathbf{i}})^\dagger &= \left(X_{i_1\cdots i_{n-1}}\right)^\dagger \delta_{li_n} \qquad \text{(by (20))} \\ &= \sum_{\sigma\in S_1\times S_{n-2}} \left(Y_{i_{\sigma(1)}\cdots i_{\sigma(n-1)}}\right)^\dagger c_{i_{\sigma(1)}\cdots i_{\sigma(n-1)},i_1\cdots i_{n-1}}\delta_{li_n} \qquad \text{(by (39))}. \end{aligned}$$

The ${}_l\partial$ applied to the right-hand side of (39) gives

$$\sum_{\pi\in S_1\times S_{n-1}} {}_l\partial(Y_{\mathbf{i}\cdot\pi})^\dagger c_{\mathbf{i}\cdot\pi,\mathbf{i}} = \sum_{\pi\in S_1\times S_{n-1}} \sum_{(n\geqslant j\geqslant 2;l=\pi(j))} d^{(j)}_{\mathbf{i}\cdot\pi}\left(Y_{i_{\pi(1)}\cdots\widehat{i_{\pi(j)}}\cdots i_{\pi(n)}}\right)^\dagger c_{\mathbf{i}\cdot\pi,\mathbf{i}} \qquad \text{(by (37))}.$$

By linear independence of $Y_{\mathbf{i}}$ (cf corollary 1) we obtain the following system of $(n-1)!$ equations (in the generic case) for $(n-1)!$ unknown coefficients $c_{\mathbf{i}\cdot\pi,\mathbf{i}}$ ($i_1 = k, \pi\in S_1\times S_{n-1}$):

Equations for $c_{\mathbf{j},\mathbf{i}}$

$$\sum_{n\geqslant j\geqslant 2} d^{(j)}_{\mathbf{i}\cdot\pi t_{j,n}} c_{\mathbf{i}\cdot\pi t_{j,n},\mathbf{i}} = \delta_{\pi(n),n} c_{(\mathbf{i}\cdot\pi)',\mathbf{i}'} \tag{40}$$

where $\pi\in S_1\times S_{n-1}$, $t_{j,n}$ denotes the cyclic permutation which sends $1, 2, \ldots, j, j+1, \ldots, n$ to $1, 2, \ldots, n, j, \ldots, n-1$ and $\mathbf{i}' = i_1\ldots i_{n-1}$.

Note that our derivation of the equations (40) (generic case) will yield (by summation) the equations for the nongeneric case (i.e. when there are repetitions among $i_1, \cdots, i_n$). This justifies the form of our expression (15) for the number operators N_k.

The second approach to solving the recursive system (20) for $X_{\mathbf{i}}$ is to write $Y_{\mathbf{i}}$ in terms of $X_{\mathbf{i}}$, again with the first index fixed ($= k$ for N_k).

$$(Y_{\mathbf{i}})^\dagger = \sum_{\mathbf{j}=\mathbf{i}\cdot\pi,\pi\in S_1\times S_{n-1}} (X_{\mathbf{j}})^\dagger e_{\mathbf{j},\mathbf{i}}. \tag{41}$$

Proposition 3. *The coefficients* $e_{\mathbf{j},\mathbf{i}}$ *satisfy the following recursions:*

$$e_{\mathbf{i}\cdot\pi,\mathbf{i}} = d^{(r)}_{\mathbf{i}} e_{\mathbf{i}'\cdot\pi',\mathbf{i}'} \tag{42}$$

where $r = \pi(n)$, $\mathbf{i}' = i_1\ldots i_{n-1}$, $\pi' = t_{r,n}\pi$ $(\Rightarrow \pi = t^{-1}_{r,n}\pi', \pi'\in S_{n-1})$, *and* $d^{(r)}_{\mathbf{i}} = d^{(r)}_{i_1\cdots i_n}$ *is defined in (38).*

Proof. By applying ${}_l\partial$ to both sides of (41), and using (37), we obtain

$$\sum_{\mathbf{j}=\mathbf{i}\cdot\pi,\pi\in S_1\times S_{n-1}} \left(X_{j_1\ldots j_{n-1}}\right)^\dagger e_{\mathbf{j},\mathbf{i}}\delta_{l,j_n} = \sum_{r\geqslant 2,i_r=l} d^{(r)}_{\mathbf{i}}\left(Y_{i_1\ldots\hat{i}_r\ldots i_n}\right)^\dagger \tag{43}$$

$$= \sum_{r\geqslant 2,i_r=l} d^{(r)}_{\mathbf{i}} \sum_{\sigma\in S_1\times S_{n-2}} (X_{\mathbf{i}_{\hat{r}}\cdot\sigma})^\dagger e_{\mathbf{i}_{\hat{r}}\cdot\sigma,\mathbf{i}_{\hat{r}}} \tag{44}$$

where $\mathbf{i}_{\hat{r}} := i_1\ldots i_{r-1}i_{r+1}\ldots i_n$. Observe that $i_{\pi(1)}\ldots i_{\pi(n-1)} = \mathbf{i}_{\hat{r}}\cdot\sigma$ iff $r = \pi(n)$ and $\sigma = t_{r,n}\pi$. By equating the coefficients in (43) and (44) the proof of proposition 3 follows. $\square$

Note that the recursion (42) corresponds to the multiplication by the following element (of the twisted group algebra):

$$\eta_n := \sum_{k=2}^{n} Q_{\{k,k+1\}} \cdots Q_{\{k,n\}}(1 - Q_{\{k,1\}} \cdots Q_{\{k,k-1\}})\tilde{t}_{k,n}^{-1}. \tag{42a}$$

Let $E = (e_{\mathbf{i},\mathbf{j}})$, with $i_1 = j_1 (=k)$ fixed, be the $(n-1)! \times (n-1)!$ transition matrix (in the generic case), with entries $e_{\mathbf{i},\mathbf{j}}$ from (41). In [8], the linear equations for the entries of E^{-1} are constructed for general n and solved in special cases for $n = 1, 2, 3$. From these computations it was conjectured (in [8]) that E^{-1} is related to the inverse of the Gram matrix A, see equation (3); here we prove this conjecture.

By comparing ξ_n from (36) with η_n from (42a) we get

$$w_n \eta_n w_n = \xi_n$$

and deduce the following:

Lemma 2. *The matrix E is the matrix of the right multiplication by the following element of our twisted group algebra $K_n\tilde{\ }[S_n]$:*

$$w_n \widetilde{\alpha}_{n-1,1} \widetilde{\delta}_n w_n. \tag{45}$$

Here $w_n = n \ldots 21$ denotes the longest element in S_n.

Proof. It follows by iteratively applying the result of lemma 1, using the definition (32) of $\tilde{\delta}_n$ together with the recursions obtained in proposition 3. □

5. The main results

Now we prove the following theorem:

Theorem 1. *The number operators in the multiparameter quon algebra $\mathcal{A}^{(\mathbf{q})}$ equation (1) are given, in the expanded form, by*

$$N_k = a_k^\dagger a_k + \sum_{n=1}^{\infty} \sum_{\mathbf{i}, i_1 = k} \sum_{\pi \in S_1 \times S_{n-1}} \hat{A}_{\mathbf{i},\mathbf{i}\cdot\pi}^{-1} (Y_{\mathbf{i}\cdot\pi})^\dagger Y_{\mathbf{i}} \tag{46}$$

where the matrix $\hat{A}$ denotes the matrix obtained from the Gram matrix $A = \oplus_{n\geqslant 0} \oplus_{k_1 \leqslant \cdots \leqslant k_n} A^{k_1 \ldots k_n}$ (described in (4)) by replacing each block $A^{k_1 \ldots k_n}$ $(k_1 \leqslant \cdots \leqslant k_n)$ with a specialized $n! \times n!$ block $A^{12\cdots n}|_{1\mapsto k_1, 2\mapsto k_2 \cdots n \mapsto k_n}$ and $Y_{\mathbf{i}}$ are given by (22).

Or, in the reduced form, by

$$N_k = a_k^\dagger a_k + \sum_{n=1}^{\infty} \sum_{\mathbf{i}, i_1 = k} \sum_{\tilde{\pi} \in Stab_{\mathbf{i}} \backslash S_1 \times S_{n-1}} \tilde{A}_{\mathbf{i},\mathbf{i}\cdot\tilde{\pi}}^{-1} (Y_{\mathbf{i}\cdot\tilde{\pi}})^\dagger Y_{\mathbf{i}} \tag{47}$$

where the reduction procedure is given with respect to the groups $S_1 \times S_{n-1}$ (instead of S_n) analogously to the reduction procedure described in the text preceding (6).

The proof of this theorem relies on one more lemma.

Lemma 3. *We have*

The $S_{1,n-2,1}$-component of $\tilde{\alpha}_n{}^{-1}$ = the $S_{1,n-2,1}$-component of $\widetilde{\delta}_n^{-1} \times \widetilde{\alpha}_{n-1,1}^{-1}$.

Proof of lemma 3. This is a generalization of a Zagier result [15]. Here we sketch the proof. By observing that $\widetilde{\alpha}_{n-1} = \widetilde{\alpha}_{n-1,1}$ we can write (cf (33))

$$\widetilde{\alpha}_n = \widetilde{\alpha}_{n-1,1}\widetilde{\delta}_n\widetilde{\gamma}_n^{-1} \qquad \widetilde{\alpha}_n^{-1} = \widetilde{\gamma}_n\widetilde{\delta}_n^{-1}\widetilde{\alpha}_{n-1,1}^{-1} \tag{48}$$

where, according to (31),

$$\tilde{\gamma}_n = (1-\widetilde{t}_{1,n})(1-\widetilde{t}_{2,n})\cdots(1-\widetilde{t}_{n-1,n}) = \sum_{k=1}^{n}(-1)^{n-k}\sum_{\pi\in S_{n,k}}\tilde{\pi}^{-1} \tag{49}$$

with $S_{n,k} \subset S_n$ denoting the set of all permutations such that $\pi(1) < \cdots < \pi(k) = n > \cdots > \pi(n)$. Note that $\widetilde{\delta}_n$ involves only permutations belonging to $S_{n-1} \times S_1$ (cf (32); for an explicit formula for the inverse of $\widetilde{\delta}_n$ see proposition 2.1.1 in [10]). Now it is clear that only the trivial term in $\tilde{\gamma}_n$ can contribute to the $S_{1,n-2,1}$-component of $\tilde{\alpha}_n^{-1}$. Lemma 3 is proved. This establishes the connection between E^{-1} and the inverse A^{-1} of the Gram matrices. □

Proof of theorem 1. By using lemmas 2 and 3, together with the symmetry property (9) and Hermiticity (8) of the multiparameter Zagier matrices, we obtain

$$X_{\mathbf{i}}^{\dagger} = \sum_{\pi\in S_1\times S_{n-1}} Y_{\mathbf{i}\cdot\pi}^{\dagger} A_{\mathbf{i},\mathbf{i}\cdot\pi}^{-1}$$

in expanded form, and similarly

$$X_{\mathbf{i}}^{\dagger} = \sum_{\tilde{\pi}\in H\backslash S_1\times S_{n-1}} Y_{\mathbf{i}\cdot\tilde{\pi}}^{\dagger} A_{\mathbf{i},\mathbf{i}\cdot\tilde{\pi}}^{-1}$$

in reduced form. This completes the proof of theorem 1. The method for calculating the inverse of the matrix A is explained in [9, theorem 2.2.17]. □

Corollary. *Let us assume an infinite set I of indices, then the number operator N_k restricted to the finite subset $I_f \subseteq I$ is obtained from equations (46) and (47) by projecting out all words with letters from the subset I_f. In particular if $I_f = \{k\}$ we recover the simple formula for N_k for a single oscillator obtained by Greenberg [2–3].*

Also, if we plug into (46) and (47) the formulae (22) expressing $Y_{\mathbf{i}}$ in terms of monomials, we obtain Zagier or Stanciu type formulae for the number operator.

The transition operators will be considered in the near future.

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